

W15. Let the sequences $(a_n)_{n \geq 1}$, $(b_n)_{n \geq 1}$:

$$a_n = \sum_{k=1}^n \arctan \frac{1}{k^2 - k + 1} \quad \text{and} \quad \lim_{n \rightarrow \infty} \frac{b_{n+1}}{nb_n} = b \in \mathbb{R}_+^*.$$

Compute:

$$\lim_{n \rightarrow \infty} \left(\frac{\pi}{2} - a_n \right) \sqrt[n]{b_n}$$

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Since $\arctan \frac{1}{k^2 - k + 1} = \arctan \frac{1}{k-1} - \arctan \frac{1}{k}$ (because

$$\tan \left(\arctan \frac{1}{k-1} - \arctan \frac{1}{k} \right) = \frac{\frac{1}{k-1} - \frac{1}{k}}{1 + \frac{1}{k-1} \cdot \frac{1}{k}} = \frac{1}{k^2 - k + 1}) \text{ then}$$

$$a_n = \sum_{k=1}^n \arctan \frac{1}{k^2 - k + 1} = \sum_{k=1}^n \left(\arctan \frac{1}{k-1} - \arctan \frac{1}{k} \right) = \frac{\pi}{2} - \arctan \frac{1}{n}$$

and, therefore, $\lim_{n \rightarrow \infty} \left(\frac{\pi}{2} - a_n \right) \sqrt[n]{b_n} = \lim_{n \rightarrow \infty} \arctan \frac{1}{n} \sqrt[n]{b_n} =$

$$\lim_{n \rightarrow \infty} \left(n \arctan \frac{1}{n} \cdot \right) = \lim_{n \rightarrow \infty} \frac{\sqrt[n]{b_n}}{n} = \lim_{n \rightarrow \infty} \sqrt[n]{\frac{b_n}{n^n}}.$$

Since $\lim_{n \rightarrow \infty} \left(\frac{b_{n+1}}{(n+1)^{n+1}} \div \frac{b_n}{n^n} \right) = \lim_{n \rightarrow \infty} \left(\frac{b_{n+1}}{b_n} \cdot \frac{n^n}{(n+1)^{n+1}} \right) =$

$$\lim_{n \rightarrow \infty} \left(\frac{b_{n+1}}{nb_n} \cdot \frac{1}{\left(1 + \frac{1}{n}\right)^{n+1}} \right) = e \cdot b$$